Invariant measures of the 2D Euler equations and applications to equilibrium and non equilibrium phase transitions

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In 1949, L. Onsager first suggested to use statistical mechanics in order to understand the self organization of two dimensional and geophysical flows (atmospheres, oceans) into large scale structures (jets and vortices). However, this wonderful insight faced huge theoretical difficulties: the mathematical tools in order to properly understand two dimensional turbulence in the equilibrium statistical framework were not available (it was not understood how to define the equilibrium statistical mechanics of fields, related to the Rayleigh-Jeans paradox). Using the point vortex model, a finite dimensional approximations of the two-dimensional Euler equations, allowed L. Onsager to by-pass this problem. This pioneer work started the statistical mechanics of two dimensional turbulent flows. Since then, this theory has been wonderfully developed both by physicists and mathematicians, for instance leading to the proof that a mean field approach is exact in the limit of an infinite number of vortices. In the introduction, I will briefly review some of the recent results about the point vortex model and its statistical properties.

The main aim of this talk is to overview recent development of Onsager ideas beside the point vortex model (equilibrium statistical mechanics of the 2D Euler equations, and non equilibrium statistical mechanics of the 2D-Stochastic Navier-Stokes equations) and their applications to model physical phenomena (the Great Red Spot of Jupiter, ocean jets and vortices, equilibrium and non-equilibrium phase transitions).

The equilibrium statistical mechanics of the 2D Euler equations, the Robert-Sommeria-Miller theory, will be briefly described. I will sketch a recent proof that microcanonical measures of the 2D Euler equations are actually invariant measures for the dynamics. I will also explain how it is possible to explicitly built sets of invariant measures that generalize statistical equilibrium measures for the two dimensional Euler and the Vlasov equations, proving that these equations are not ergodic.

The theory of phase transitions for the 2D Euler equations, has been the subject of many recent works: a classification of phase transitions, existence of non-equivalence between the microcanonical and the canonical ensembles. Some of these results will be illustrated through the numerical sampling of microcanonical measures using Creutz algorithm, and through applications to physical phenomena.

For the two-dimensional Navier-Stokes equations with weak stochastic forcing and dissipation, the existence of an invariant measure has been mathematically proved recently, together with mixing and ergodic properties. I will sketch how to use the invariant measures of the two dimensional Euler equations to describe self-consistently the invariant measures for the two dimensional Navier-Stokes equations. We predict for instance non-equilibrium phase transitions, and observe them in numerical experiments, and in laboratory experiments.

Onsager was also the first to understand that the time reversible 3D Euler equations have an irreversible behavior and are able to dissipate energy due to a lack of regularity of the flow. I will describe new mathematical results about the inviscid relaxation of the two dimensional Euler equations (explicit predictions of the large time asymptotics). These results are one of the few examples in statistical mechanics where the apparent paradox between microscopic reversibility and macroscopic irreversibility can be understood and analyzed thoroughly theoretically.

References

[1] F. Bouchet and A. Venaille, 2011, *Statistical mechanics of two-dimensional and geophysical flows,* **to be published in Physics Reports** (http://perso.ens-lyon.fr/freddy.bouchet/Publications/Physics-Report-v3.pdf)