Kinetic theory of two-dimensional point vortices

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In this seminar, I will discuss different aspects of the kinetic theory of two-dimensional point vortices.

In most of the seminar, I will consider "standard" point vortices whose dynamical evolution is described by the Kirchhoff-Hamilton equations

$$\gamma\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i},\qquad \gamma\frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i},\tag{1}$$

$$H=\gamma^2\sum_{i<j}u(|\mathbf{r}_i-\mathbf{r}_j|).\tag{2}$$

For the simplicity of the presentation, I will assume that the vortices have the same circulation γ. On the other hand, the potential of interaction $u(|\mathbf{r}-\mathbf{r}'|)$ will be assumed to satisfy the Poisson equation $\Delta u=-\delta$. More general situations can be contemplated. This N-vortex system is a physically motived model whose statistical mechanics was pioneered by Onsager in 1949. This model of point vortices is associated with the *microcanonical ensemble* in which the energy is fixed. I will consider the thermodynamic limit where the number of point vortices $N\to+\infty$ while their individual circulation $\gamma\sim 1/N\to 0$ and the area of the domain $A\sim 1$. For $t\to+\infty$, the smooth vorticity field $\omega(\mathbf{r},t)$ is expected to reach the Boltzmann distribution $\omega(\mathbf{r})=-\Delta\psi=Ae^{-\beta\gamma\psi_*(\mathbf{r})}$ which maximizes the Boltzmann entropy $S=-\int\frac{\omega}{\gamma}\ln\frac{\omega}{\gamma}\,d\mathbf{r}$ at fixed circulation $\Gamma=N\gamma=\int\omega\,d\mathbf{r}$, energy $E=\frac{1}{2}\int\omega\psi\,d\mathbf{r}$ and angular momentum $L=\int\omega r^2\,d\mathbf{r}$. However, in order to determine whether the system truly reaches the Boltzmann distribution, and to obtain the scaling with N of the relaxation time, we must develop a kinetic theory of point vortices.

In the first part of the seminar, I consider the evolution of the system "as a whole". When collective effects are neglected, the evolution of the smooth vorticity field $\omega(\mathbf{r},t)$ is described by a general kinetic equation

$$\begin{aligned}\frac{\partial\omega}{\partial t}+\frac{N-1}{N}\mathbf{u}\cdot\nabla\omega&=\frac{\partial}{\partial r^\mu}\int_0^t d\tau\int d\mathbf{r}_1 V^\mu(1\to 0)\mathcal{G}(t,t-\tau)\\&\times\left[\mathcal{V}^\nu(1\to 0)\frac{\partial}{\partial r^\nu}+\mathcal{V}^\nu(0\to 1)\frac{\partial}{\partial r_1^\nu}\right]\omega(\mathbf{r},t-\tau)\frac{\omega}{\gamma}(\mathbf{r}_1,t-\tau),\end{aligned}\tag{3}$$

which is valid for flows that are not necessarily axisymmetric and not necessarily Markovian. This kinetic equation can be derived from projection operator methods, from the BBGKY hierarchy or from a quasilinear theory. It is valid at the order $1/N$ in the thermodynamic limit. For $N\to+\infty$, the collisions (correlations) between point vortices can be neglected. In that case, the mean field approximation becomes exact and one obtains the 2D Euler-Poisson system

$$\frac{\partial\omega}{\partial t}+\mathbf{u}\cdot\nabla\omega=0,\qquad \mathbf{u}=-\mathbf{z}\times\nabla\psi,\qquad \Delta\psi=-\omega,\tag{4}$$

describing the "collisionless" evolution of the point vortex gas. The kinetic equation (3), taking into account two-body distant collisions between vortices, represents the $1/N$ correction to the 2D Euler equation. It describes the collisional evolution of the point vortex gas on a timescale of order Nt_D, where t_D is the dynamical time.

If we consider Euler stable axisymmetric distributions $\omega(\mathbf{r},t)=\omega(r,t)$ and make a Markovian approximation (which is generally valid for $N\gg 1$), the kinetic equation (3) reduces to the form

$$\frac{\partial\omega}{\partial t}=2\pi^2\gamma\frac{1}{r}\frac{\partial}{\partial r}\int_0^{+\infty}r'\,dr'\,\chi(r,r',t)\delta[\Omega(r,t)-\Omega(r',t)]\left(\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r'}\frac{\partial}{\partial r'}\right)\omega(r,t)\omega(r',t),\tag{5}$$

where

$$\chi(r,r')=\sum_n|n|\hat{u}_n(r,r')^2.\tag{6}$$

This kinetic equation conserves the circulation Γ, the energy E and the angular momentum L. The Boltzmann entropy S satisfies $\dot{S}\geq 0$. If the profile of angular velocity $\Omega(r,t)$ is monotonic, the collision term (r.h.s.) vanishes because there is no resonance between point vortices. In that case, there is no evolution on the timescale $\sim Nt_D$ and the collisional evolution is due to more complex (three-body, four-body,...) correlations. This implies that, for axisymmetric distributions with monotonic profile of angular velocity, the relaxation time towards the Boltzmann distribution is $t_R > Nt_D$. However, for flows that are non-axisymmetric, it can be of the order $t_R \sim Nt_D$ due to the occurence of additional resonances. When collective effects are taken into account, the kinetic equation takes the same form as Eq. (5) except that the bare potential of interaction is replaced by a "dressed" potential of interaction arising from the polarization of the medium. However, the main results of the simplified kinetic theory (neglecting collective effects) are not altered.

In the second part of the seminar, I consider the relaxation of a test vortex in a bath of field vortices. The probability density $P(\mathbf{r},t)$ of finding the test vortex in $\mathbf{r}$ at time t is solution of a Fokker-Planck equation

$$\frac{\partial P}{\partial t} = 2\pi^2\gamma\frac{1}{r}\frac{\partial}{\partial r}\int_0^{+\infty} r'\,dr'\,\chi(r,r')\delta[\Omega(r)-\Omega(r')]\left(\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r'}\frac{\partial}{\partial r'}\right)P(r,t)\omega(r'), \tag{7}$$

incorporating a term of diffusion

$$D = 2\pi^2\gamma\frac{1}{r^2}\int_0^{+\infty} r'\,dr'\,\chi(r,r')\delta[\Omega(r)-\Omega(r')]\omega(r'), \tag{8}$$

and a term of drift

$$V_{drift} = 2\pi^2\gamma\frac{1}{r}\int_0^{+\infty} dr'\,\chi(r,r')\delta[\Omega(r)-\Omega(r')]\frac{d\omega}{dr}(r'). \tag{9}$$

The test particle relaxes towards the distribution of the bath on a timescale $t_R^{bath} \sim (N/\ln N)t_D$. For an out-of-equilibrium distribution of the bath $\omega(r)$, which can be any stable steady state of the 2D Euler equation with a monotonic profile of angular velocity, the Fokker-Planck equation can be written

$$\frac{\partial P}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD(r)\left(\frac{\partial P}{\partial r} - P\frac{d\ln\omega}{dr}\right)\right]. \tag{10}$$

The diffusion coefficient and the drift term are given by

$$D(r) = 2\pi^2\gamma\frac{\chi(r,r)}{|\Sigma(r)|}\omega(r), \qquad V_{drift} = 2\pi^2\gamma\frac{\chi(r,r)}{|\Sigma(r)|}\frac{d\omega}{dr}(r), \qquad V_{drift} = D(r)\frac{d\ln\omega}{dr}, \tag{11}$$

where $\Sigma(r) = r\Omega'(r)$ is the local shear. The diffusion coefficient is proportional to the vorticity field and inversely proportional to the shear. The drift term is proportional to the gradient of the vorticity field and inversely proportional to the shear. For a thermal bath $\omega(r) = Ae^{-\beta\gamma\psi_*(r)}$, the Fokker-Planck equation takes the form

$$\frac{\partial P}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD(r)\left(\frac{\partial P}{\partial r} + \beta\gamma P\frac{d\psi_*}{dr}\right)\right]. \tag{12}$$

The drift term is given by

$$V_{drift} = -D\beta\gamma\frac{d\psi_*}{dr}. \tag{13}$$

The drift coefficient (mobility) is related to the diffusion coefficient by an Einstein relation

$$\xi = D\beta\gamma. \tag{14}$$

Some properties of the Fokker-Planck equation (7) will be given during the seminar. The kinetic theory can be extended to the case of point vortices with different circulations.

I will also (briefly) discuss a rather academic model of "Brownian vortices" which is associated with the *canonical ensemble* in which the temperature is prescribed. In that case, the deterministic Hamilton-Kirchoff equations (1)-(2) are replaced by stochastic Langevin equations

$$\frac{d\mathbf{r}_i}{dt} = -\frac{1}{\gamma}\mathbf{z}\times\nabla_i H - \nu\beta\nabla_i H + \sqrt{2\nu}\mathbf{R}_i(t), \tag{15}$$

where $\mathbf{R}_i(t)$ is a white noise such that $\langle \mathbf{R}_i(t)\rangle = \mathbf{0}$ and $\langle R_i^\alpha(t) R_j^\beta(t')\rangle = \delta_{ij}\delta_{\alpha\beta}\delta(t-t')$. In the thermodynamic limit $N\rightarrow +\infty$, the evolution of the smooth vorticity field is governed by a mean field Fokker-Planck equation of the form

$$\frac{\partial\omega}{\partial t}+\mathbf{u}\cdot\nabla\omega=\nu\nabla\cdot(\nabla\omega+\beta\gamma\omega\nabla\psi), \qquad -\Delta\psi=\omega. \tag{16}$$

This equation exhibits a blow-up phenomenon below a critical inverse temperature

$$\beta<\beta_c\equiv-\frac{8\pi}{N\gamma^2}. \tag{17}$$

The kinetic theory of these "Brownian vortices" can be generalized to vortices with different types of circulation. I will mention the analogy with the Smoluchowski-Poisson system describing self-gravitating Brownian particles and with the Keller-Segel model of chemotaxis.

My papers on this subject: The kinetic theory of standard point vortices is treated in [1, 2, 4, 5] (without collective effects) and in [10] (with collective effects). The kinetic theory of Brownian vortices is treated in [6]. Some reviews are given in [3, 8, 9]. The results obtained with C. Sire on the Smoluchowski-Poisson system and Keller-Segel model are summarized in [11]. A review on nonlinear mean field Fokker-Planck equations (NFP) is given in [7].

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