Variational formulation of nonlinear hydrodynamic stability

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I. OUTLINE

Nonlinear stability of fluid equilibria has been an important issue for gaining a better understanding of phenomena such as pattern formation and transition to turbulence. In 1944, Landau [1] deduced a scenario about the development of a single weakly unstable mode towards (weakly) nonlinear regime, in which the nonlinear self-interaction of the dominant mode generates second harmonics and distorts the mean fields. The evolution of the dominant mode's amplitude A is possibly governed by the Landau equation, $d|A|^2/dt = 2\gamma|A|^2 - l|A|^4$, where $\gamma > 0$ is the linear growth rate and l is a constant that determines whether the linear growth will be saturated or not [2]. However, the derivation of such a greatly reduced equation (so-called the amplitude equation) from the hydrodynamic equations involves a lot of calculus and approximations. While the heuristic method have been successful in many fundamental hydrodynamic problems, it often faces technical difficulties such as "indeterminable mean field" (the equations for the wave-induced mean fields is underdetermined and requires a further assumption to fix them).

The goal of this presentation is to propose a systematic approach for deriving the amplitude equations (which describe the local bifurcations of equilibria) by means of the variational principle, namely, Hamilton's principle. In comparison to the direct method, the use of the variational principle shorten the detailed calculation of the coupling coefficients and highlights properties peculiar to dynamical systems such as the energy conservation. Moreover, the determination of the wave-induced mean fields can be carried out immediately if the equilibrium state has some spatial symmetry, as exemplified by our recent work on elliptically-strained rotating flow [3].

II. LAGRANGIAN FOR NONLINEAR DISPLACEMENT OF FLUID

In order to widen the scope of our theory as much as we can, we employ the Newcomb's Lagrangian theory [4] as the starting point, which is originally devised as the variational principle for ideal magnetohydrodynamics (MHD). We remark that this variational principle is directly applicable to inviscid compressible fluid by neglecting the magnetic field and moreover to inviscid incompressible fluid (the Euler equations) by restricting the fluid motion (and its variation as well) to be incompressible. Let us denote by $\boldsymbol{x}(t)$ the position of an infinitesimal fluid element at time t. Newcomb's Lagrangian is defined as a functional of the fluid flow map $\varphi_t : \boldsymbol{x}(0) \in D \to \boldsymbol{x}(t) \in D$, which is mathematically a one-parameter ($t \in \mathbb{R}$) diffeomorphism group on the fluid domain $D \subset \mathbb{R}^3$. The conservation laws for mass, entropy and magnetic flux are built into the variational principle as *constraints* on the Eulerian field variables, e.g., the velocity field is given by $\boldsymbol{v}(\boldsymbol{x}(t), t) = d\boldsymbol{x}/dt(t)$ and the mass density $\rho(\boldsymbol{x}(t), t)$ is also determined by $\boldsymbol{x}(t)$ if the initial data $\rho(\boldsymbol{x}(0), 0)$ is given. Now, let us regard $\boldsymbol{x}(t) = \varphi_t(\boldsymbol{x}(0))$ as unperturbed fluid motion and denote the perturbed motion by $\boldsymbol{x}(t) + \Xi(\boldsymbol{x}(t), t)$, where the displacement map $\Xi(\boldsymbol{x}, t)$ is again a diffeomorphism. By expanding Newcomb's Lagrangian with respect to Ξ around a given equilibrium state, we obtain the Lagrangian for $\Xi(\boldsymbol{x}, t)$ in the form of

$$L[\Xi] = \int \frac{\rho}{2} \left| \frac{D\Xi}{Dt} \right|^2 d^3x - \frac{W^{(2)}(\Xi, \Xi)}{2} - \frac{W^{(3)}(\Xi, \Xi, \Xi)}{3!} - \frac{W^{(4)}(\Xi, \Xi, \Xi, \Xi)}{4!} - \dots \tag{1}$$

where $D/Dt = \partial/\partial t + \boldsymbol{v} \cdot \nabla$ (ρ and $\boldsymbol{v}$ are, respectively, equilibrium density and velocity). We formally refer to $W^{(n)}$ as nth-order potential energy, which does not include the time derivative of Ξ. There is some history about the derivation of the general expression (1). In the limit of infinitesimally small Ξ, one can neglect $W^{(n)}$, $n \geq 3$, and reproduce the quadratic Lagrangian derived by Dewar [5]. By introducing a linear operator $\mathcal{F}$ by $W^{(2)} = -\int \Xi \cdot \mathcal{F}\Xi d^3x$ (which turns out to be symmetric), the Euler-Lagrange equation agrees with the well-known linearized equation of motion $\rho D^2\Xi/Dt^2 = \mathcal{F}\Xi$ [6]. Existence of this variational formulation of the linearized systems has been noted by many pioneering works [7–9] and utilized extensively in linear stability analysis. However, the generalization to nonlinear (or finite amplitude) displacement is not so straightforward since Ξ should not be regarded as a vector field, but as a diffeomorphism. Pfirsch & Sudan [10] derived $W^{(3)}$ in the absence of equilibrium flow ($\boldsymbol{v} = 0$) and showed a few special examples of equilibria at which $W^{(3)}$ vanishes for marginal modes. The author [11] recently find out a more complete form of $W^{(3)}$ with cubic symmetry, including flow ($\boldsymbol{v} \neq 0$), where the application of the Lie series expansion is a key technique for finding symmetric forms of $W^{(n)}$.

III. AMPLITUDE EQUATIONS FOR WAVE-WAVE INTERACTIONS

Once the Lagrangian for the displacement map Ξ is formulated as shown in (1), we can derive the amplitude equations in a systematic way. Since the Euler-Lagrange equation is written as

$$\rho \frac{D^2\Xi}{Dt^2} = \mathcal{F}\Xi + \frac{1}{2}\mathcal{F}^{(2)}(\Xi, \Xi) + \frac{1}{3!}\mathcal{F}^{(3)}(\Xi, \Xi, \Xi) + \dots, \tag{2}$$

nonlinear forces $\mathcal{F}^{(n)}(\dots)$, $n \geq 2$, defined by $W^{(n)}(\dots) = -\int \Xi \cdot \mathcal{F}^{(n)}(\dots) d^3x$, are responsible for nonlinear wave-wave interactions. If there exist three waves satisfying the resonance conditions for frequencies, $\omega_a + \omega_b + \omega_c = 0$, and for wave numbers, $k_a + k_b + k_c = 0$, this triad will be coupled via the quadratic force $\mathcal{F}^{(2)}$. The application of the averaged Lagrangian method [12] leads immediately to the amplitude equations for the three-wave resonance with appropriate Hamiltonian property [11]. In doing so, one needs neither to invoke the solvability condition nor to spend any effort on proving the energy conservation among the resonant triad.

If the three-wave resonance is absent or ineffective, the four-wave resonance via the cubic force $\mathcal{F}^{(3)}$ becomes dominant as the next nonlinearity. In particular, a linearly unstable mode will be subject to the self-interaction due to $\mathcal{F}^{(3)}$ as well as the non-resonant interaction with the second harmonics due to $\mathcal{F}^{(2)}$. In this case, one should put $\Xi = A\hat{\boldsymbol{\xi}}_k + A^2\hat{\boldsymbol{\xi}}_{2k}^{(2)} + |A|^2\hat{\boldsymbol{\xi}}_0^{(2)} + \text{c.c.}$ with $\hat{\boldsymbol{\xi}}_k$ being a linearly unstable eigenmode and then solve the second harmonics $\hat{\boldsymbol{\xi}}_{2k}^{(2)}$ and the mean displacement $\hat{\boldsymbol{\xi}}_0^{(2)}$. However, in this Lagrangian approach, there is often no need to solve $\hat{\boldsymbol{\xi}}_0^{(2)}$ since $\hat{\boldsymbol{\xi}}_0^{(2)} = 0$ follows automatically from the spatial symmetry of the equilibrium state. As a result, we can derive the amplitude equation for A according to the variational principle and show that it conforms to the Landau equation when the linearly unstable mode is a propagating wave relative to the mean flow. On the other hand, if the fundamental mode is not propagating (i.e., the static bifurcation), the amplitude equation is reduced to a second-order equation $d^2A/dt^2 = \gamma^2 A - l' A^3$ with a different formula for the constant l'.

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