Bifurcation structure of the mean field equation for an annular domain

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We consider the mean field equation with a definite-signed intensity:

$$\text{(MF)} \begin{cases} -\Delta u = \lambda \frac{e^u}{\int_\Omega e^u dx} & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where λ is a positive parameter and Ω_ε is a two-dimensional annulus defined by

$$\Omega_\varepsilon := \left\{ x \in \mathbb{R}^2;\, \varepsilon < |x| < 1 \right\}$$

for $0 < \varepsilon < 1$. The aim of this talk is to determine the solution structure of (MF).

Nagasaki-Suzuki [3] and Lin [2] studied the Gel'fand problem which is related to the problem (MF):

$$\text{(G)} \begin{cases} -\Delta u = \rho\, e^u & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

and they independently showed that radial solutions of (G) generates a branch in ρ–u plane and the solution blows up on some circle as $\rho \to 0$. This branch corresponds to the solution branch of (MF) for $\lambda > 0$. Moreover, Lin showed that the radial branch has infinitely many bifurcation points from which non-radial solutions emanates and Nagasaki-Suzuki also obtained non-radial solutions such that the quantity $\int_\Omega e^u dx$ is large. After that Dancer [1] showed that the non-radial branches obtained by Lin are unbounded in ρ–u plane. These facts indicate that the solution set of (G) is as illustrated in Figure 1, and the corresponding diagram for (MF) is Figure 2.

The goal of this study is to verify that these figures are correct for small ε. To this end, we consider what is the appropriate limit of (MF) as $\varepsilon \to 0$. After an appropriate scaling, we obtain the following limiting equation:

$$-\Delta u = A \frac{e^u}{\int_{\mathbb{R}^2} e^u dx} \quad \text{in } \mathbb{R}^2 \setminus \{0\},$$

and the boundary condition:

$$u(x) = (B-2)\log|x| + o(1) \quad \text{as } |x| \to 0,$$
$$u(x) = -(B+2)\log|x| + o(1) \quad \text{as } |x| \to \infty.$$

Here $A > 0$ and $B \geq 2$ are parameters. We can check that the following functions are solutions of the above problem:

$$U = U_K(x) = \log \frac{1}{r^2(r^K + r^{-K})^2} \quad \text{with } A = 8K\pi, B = 2K, \ (K \geq 1),$$

$$V = V_{k,a,\gamma}(x) = \log \frac{1}{r^2\{r^k + r^{-k} - 2(1-a)^{1/2}\cos(k\theta + \gamma)\}^2}$$
$$\text{with } A = 8k\pi, B = 2k \quad (k \in \mathbb{N}, 0 < a < 1, \gamma \in \mathbb{R}),$$

where $x = (r\cos\theta, r\sin\theta)$.

By using the above functions, we can construct non-radial solutions of (MF).

Theorem 1. *For any $k = 2, 3, \ldots$ and $0 < a < 1$, there is ε_0 and a family of solutions $\{(\lambda_\varepsilon, u_{\varepsilon,\gamma})\}_{0<\varepsilon<\varepsilon_0, 0\leq\gamma<2\pi}$ of (MF) such that*

$$\lambda_\varepsilon = 8k\pi + O\left(\varepsilon^{k-1}\log\frac{1}{\varepsilon}\right),$$

$$u_{\varepsilon,\gamma}(r,\theta) = \left(k - \frac{1}{k}\right)\log\frac{1}{\varepsilon} + V(\varepsilon^{\frac{k-1}{2k}} r, \theta) + O\left(\varepsilon^{k-1}\log\frac{1}{\varepsilon}\right) \quad \textit{in } L^\infty$$

as $\varepsilon \to 0$.

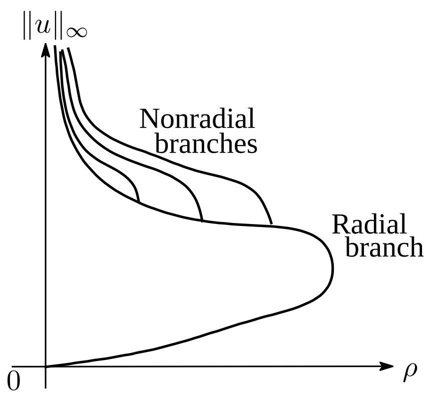

Figure 1 : Bifurcation diagram for (G)

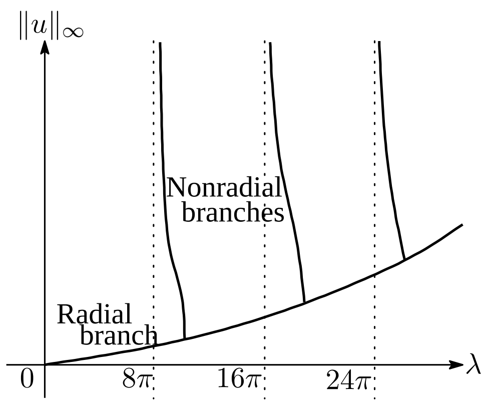

Figure 2 : Bifurcation diagram for (MF)

References

[1] E.N. Dancer, *Global breaking of symmetry of positive solutions on two-dimensional annuli*, Differential Integral Equations **5** (1992), 903–913.

[2] S.-S. Lin, *On non-radially symmetric bifurcation in the annulus*, J. Differential Equations **80** (1989), 251–279.

[3] K. Nagasaki and T. Suzuki, *Radial and nonradial solutions for the nonlinear eigenvalue problem $\Delta u + \lambda e^u = 0$ on annuli in $\mathbb{R}^2$*, J. Differential Equations **87** (1990), 144–168.