To what extent can the Hamiltonian of vortices illustrate the mean field of equilibrium vortices ?

as a substitution of "Opening Talk ; Purpose of this seminar[1]"

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1 Introduction

In this talk, we are concerned with *the Gel'fand problem* in two space dimensions:

$$-\Delta u = \lambda e^u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $\lambda > 0$ is a parameter. Let $\{\lambda_n\}_{n\in\mathbf{N}}$ be a sequence satisfying $\lambda_n \downarrow 0$ and $u_n = u_n(x)$ be a solution to (1) for $\lambda = \lambda_n$. The fundamental facts concerning the asymptotic behavior of u_n are established by the pioneering work of Nagasaki and Suzuki:

Fact 1 ([14])**.** Let $\Sigma_n = \lambda_n \int_\Omega e^{u_n}$. Then $\{\Sigma_n\}$ accumulate to Σ_∞ which is either

(i) 0, (ii) $8\pi m$ ($m \in \mathbf{N}$), or (iii) $+\infty$.

According to these cases, the (sub-)sequence of solutions $\{u_n\}$ behave as follows:

(i) uniform convergence to 0,

(ii) m-point blow-up, that is, there is a blow-up set $\mathscr{S} = \{\kappa_1, \dots, \kappa_m\} \subset \Omega$ of distinct m-points such that

$$u_n \longrightarrow u_\infty(x) = 8\pi \sum_{j=1}^m G(x, \kappa_j) \quad \text{locally uniformly,} \tag{2}$$

where $G(x, y)$ is the Green function of $-\Delta$ under the Dirichlet condition, that is,

$$-\Delta G(\cdot, y) = \delta_y \quad \text{in } \Omega, \quad G(\cdot, y) = 0 \quad \text{on } \partial\Omega.$$

(iii) entire blow-up, that is, $u_n(x) \longrightarrow +\infty$ for every $x \in \Omega$.

Moreover, the blow-up points $\kappa_i (i = 1, \cdots, m)$ in case (ii) satisfy the relations

$$\nabla \left(K(x, \kappa_i) + \sum_{1\le j\le m,\, j\ne i} G(x, \kappa_j) \right)\Bigg|_{x=\kappa_i} = 0, \tag{3}$$

where $K(x, y) = G(x, y) - \frac{1}{2\pi} \log |x - y|^{-1}$.

$K(x, y)$ is called the regular part of the Green function $G(x, y)$. Here we set $R(x) = K(x, x)$ and introduce the function

$$H^m(x_1, \dots, x_m) = \frac{1}{2} \sum_{j=1}^m R(x_j) + \frac{1}{2} \sum_{1\le j,k\le m,\, j\ne k} G(x_j, x_k),$$

which we call *the Hamiltonian*. Since $G(x, y) = G(y, x)$ and $K(x, y) = K(y, x)$, the relation (3) means that $\mathscr{S} \in \Omega^m$ is *a critical point* of the function H^m of $2m$-variables. Therefore we are able to say that *the limit function of* $\{u_n\}$ *blows up at a critical point of the Hamiltonian* H^m. Concerning this link between H^m and $\{u_n\}$, recently we get the following result:

[1]RIMS Camp-style Seminar"Modern approach and developments to Onsager's theory on statistical vortices" at Apical Inn Kyoto (28-31, August, '11)

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Theorem 2 ([8])**.** Suppose $\mathscr{S}$ in (ii) of Theorem 1 is a *non-degenerate* critical point of H^m. Then the associated u_n for $n \gg 1$ is a *non-degenerate* critical point of the functional

$$F_{\lambda_n}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda_n \int_\Omega e^u dx.$$

It is easy to see that the equation (1) is the Euler-Lagrange equation of the functional F_λ. Therefore we may say that Theorem 2 insists deeper links between the functional F_λ and the function H^m than Theorem 1. This kind of result is sometimes called *the asymptotic non-degeneracy* and the above theorem has been already established by Gladiali and Grossi [5] for the case $m = 1$. Several studies also exist for other kind of equations (e.g., [7], [18]), but they also consider the 1-point blow-up cases. See also [6] for further correspondence concerning the Morse indices between F_λ and H^m for the case $m = 1$.

2 A short note on H^m

The Hamiltonian function H^m is rather popular in fluid mechanics. This is the Hamiltonian of *vortices* in two-dimensional incompressible non-viscous fluid.

Formally speaking, N-vortices is a set $\{(x_j(t), \Gamma_j)\}_{j=1,\cdots,N} (\subset \Omega \times (\boldsymbol{R}\backslash\{0\}))$ that forms a vorticity field $\omega(x,t) = \sum_{j=1}^N \Gamma_j \delta_{x_j(t)}$ satisfying the Euler vorticity equation

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \omega = 0, \tag{4}$$

where $\boldsymbol{v} = \nabla^\perp \int_\Omega G(x,y)\omega(y,t)dy$ is the velocity field of the fluid. Here $\nabla^\perp = \left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\right)$ and we assumed that Ω is simply connected for simplicity. δ_p is the Dirac measure supported at the point $p\,(\in \Omega)$ and Γ_j is the intensity (circulation) of the vortex at $x_j(t)$. From the Kelvin circulation law, the intensity Γ_j is considered to be conserved. From other several physical considerations, the form $\sum_{j=1}^N \Gamma_j \delta_{x_j(t)}$ is considered to be preserved during the time evolution.

It is true that the model "vortices" made many success to understand the motion of real fluid, but it should be noticed that the velocity field $\boldsymbol{v} = \sum_{j=1}^N \Gamma_j \nabla^\perp G(x, x_j(t))$ determined by the vorticity field $\sum_{j=1}^N \Gamma_j \delta_{x_j(t)}$ makes the kinetic energy $\frac{1}{2}\int_\Omega |\boldsymbol{v}|^2 dx$ infinite. Moreover it is difficult to understand, even in the sense of distributions, how it satisfies the vorticity equation (4). Nevertheless the motion of vortices have been "known" from 19th century. Indeed, they are considered to move according to the following equations:

$$\Gamma_i \frac{dx_i}{dt} = \nabla_i^\perp H^{N,\mathbf{\Gamma}}(x_1, \cdots, x_N) \left(= \left(\frac{\partial H^{N,\mathbf{\Gamma}}}{\partial x_{i,2}}, -\frac{\partial H^{N,\mathbf{\Gamma}}}{\partial x_{i,1}} \right) \right), \tag{5}$$

where

$$H^{N,\mathbf{\Gamma}}(x_1, \cdots, x_N) = \frac{1}{2}\sum_{j=1}^N \Gamma_j^2 K(x_j, x_j) + \frac{1}{2} \sum_{1 \le j,k \le N,\, j \ne k} \Gamma_j \Gamma_k G(x_j, x_k)$$

and $x_i = (x_{i,1}, x_{i,2})$. It is easy to see that the value of $H^{N,\mathbf{\Gamma}}$ is preserved under the time evolution of vortices. Therefore $H^{N,\mathbf{\Gamma}}$ is called the *Hamiltonian* of vortices.

H^m referred in Fact 1 corresponds to the special case $N = m$ and $\mathbf{\Gamma} = (\Gamma_1, \cdots, \Gamma_m) = (1, \cdots, 1)$, that is, m-vortices of *one kind*. Therefore $\mathscr{S}$ in Fact 1, that is, the possible blow-up set of the solution sequence of the Gel'fand problem is a critical point of H^m of m-vortices of *one kind*.

3 On the contrary...

It should be remarked that we are able to get the Gel'fand problem form this special Hamiltonian H^m. Indeed suppose all the intensities of vortices is equivalent to some constant Γ. Then the Hamiltonian of

m-vortices $H^{m,\mathbf{\Gamma}}$ reduces to $\Gamma^2 H^m$. In this situation, the Gibbs measure associated to this Hamiltonian is given as follows:

$$\mu^m = \frac{e^{-\tilde{\beta}\Gamma^2 H^m(x_1,\cdots,x_m)}}{\int_{\Omega^m} e^{-\tilde{\beta}\Gamma^2 H^m(x_1,\cdots,x_m)} dx_1\cdots dx_m} dx_1\cdots dx_m,$$

where $\tilde{\beta}$ is a parameter called the inverse temperature. The canonical Gibbs measure is considered in statistical mechanics to give the possibility of the state for given *energy* H^m under the fixed (inverse) temperature. If $\tilde{\beta} > 0$ (as usual), the low-energy states are likely to occur. On the contrary, if $\tilde{\beta} < 0$ (*negative* temperature cases), the high energy states have more possibility to occur, which is considered to give some reason why there are often observed large-scale long-lived structures in two-dimensional turbulence. One of the most famous example of such structures is the Jupiter's great red spot. The idea to relate such structures to negative temperature states of equilibrium vortices is first proposed by Onsager [16].

Using the canonical Gibbs measure, we are able to get the probability (density) of the first vortex observed at $x_1 \in \Omega$ from

$$\rho^m(x_1) = \int_{\Omega^{m-1}} \mu^m dx_2\cdots dx_m,$$

which is equivalent to every vortices from the symmetry of H^m. Now we assume that total vorticity is equivalent to 1, that is, $\Gamma = \frac{1}{m}$ and suppose $\tilde{\beta} = \tilde{\beta}_\infty \cdot m$ for some fixed $\tilde{\beta}_\infty \in (-8\pi, +\infty)$. Then we get ρ satisfying the following equation at the limit of ρ^m as $m \longrightarrow \infty$:

$$\rho(x) = \frac{e^{-\tilde{\beta}_\infty G\rho(x)}}{\int_\Omega e^{-\tilde{\beta}_\infty G\rho(x)} dx}, \tag{6}$$

where G is the Green operator given by $G\rho(x) = \int_\Omega G(x,y)\rho(y)dy$ ([2, Thorem 2.1]). This ρ is called the *mean field* of the equilibrium vortices of one kind. It should be remarked that when the solution of (6) is unique, ρ^m weakly converges to ρ, and not unique, to some *superposition* of ρ.

These argument was established mathmatically rigorously by Caglioti-Lions-Marchioro-Pulvirenti [2] and Kiessling [10] independently based on the argument developped by Messer-Sphon [13], see also [12]. We note that the equations similar to (6) are derived by several authors under several physically reasonable assumptions and arguments in several situations, e.g., the system of vortices of neutral and two kinds, that means there exist same numbers of vortices with positive or negative intensities with the same absolute value, was considered in [9, 17].

We also note that (6) means $u := -\tilde{\beta}_\infty G\rho$ and $\beta := -\tilde{\beta}_\infty$ satisfy

$$-\Delta u = \beta \frac{e^u}{\int_\Omega e^u\, dx} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{7}$$

Therefore each solution of (7) is linked to that of the Gel'fand problem (1) under the relation $\beta / \int_\Omega e^u dx = \lambda$, that is, $\beta = \lambda \int_\Omega e^u dx (= \Sigma)$. The behaviors of the sequences of solutions of (7) with $\beta > 0$ (that is, $\tilde{\beta}_\infty = -\beta$ is negative !) are now well studied by several authors. Especially based on the argument in [1] (see also [15]), we are able to get a subsequence satisfying $\int_\Omega e^{u_n}\, dx \longrightarrow \infty$ if $\{(u_n, \beta_n)\}$ is a sequence of solutions of (7) satisfying that $\{u_n\}$ is unbounded in $L^\infty(\Omega)$ although $\{\beta_n\}$ is bounded. Therefore, the behaviors of unbounded sequence of solutions of (7) with bounded $\{\beta_n\}$ reduce to those of (1) satisfying $\lambda_n = \beta_n / \int_\Omega e^{u_n}\, dx \longrightarrow 0$. Consequently we return to the situation of Fact 1 when we consider the mean field of equilibrium vortices with *negative* temperature and are able to represent the conclusion of Fact 1 as follows:

> The mean fields generated by equilibrium vortices of one kind with negative temperature converge only to the stationary vortices of one kind.

I consider that Theorem 2 is a next answer to the question "To what extent can the Hamiltonian of vortices illustrate the mean field of equilibrium vortices ?"

4 Sketch of the proof of Theorem 2

Similarly to [5], we prove Theorem 2 arguing by contradiction. For this purpose we assume the existence of a sequence $\{v_n\}$ of non-degenerate critical point of F_{λ_n} as $n \longrightarrow \infty$. Using the standard arguments, v_n is a non-trivial solution of the linealized problem of (1):

$$-\Delta v = \lambda_n e^{u_n} v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \tag{8}$$

Without loss of generality we may assume that $\|v_n\|_{L^\infty(\Omega)} \equiv 1$.

Taking sufficiently small $\bar{R} > 0$, we may assume that for each κ_j there exists a sequence $\{x_{j,n}\}$ satisfying

$$x_{j,n} \longrightarrow \kappa_j, \quad u_n(x_{j,n}) = \max_{B_{\bar{R}}(x_{j,n})} u_n(x) \longrightarrow \infty.$$

Then we re-scale u_n and v_n around $x_{j,n}$ as follows:

$$\widetilde{u}_{j,n}(\widetilde{x}) = u_n(\delta_{j,n}\widetilde{x} + x_{j,n}) - u_n(x_{j,n}) \quad \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0),$$
$$\widetilde{v}_{j,n}(\widetilde{x}) = v_n(\delta_{j,n}\widetilde{x} + x_{j,n}) \quad \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0),$$

where the scaling parameter $\delta_{j,n}$ is chosen to satisfy $\lambda_n e^{u_n(x_{j,n})}\delta_{j,n}^2 = 1$. From the standard argument based on the estimate concerning the blow-up behavior of u_n [11] and the classification result of the solutions of (1) and (8) in the whole space [3, 4], there exist $\boldsymbol{a}_j \in \mathbf{R}^2$, $b_j \in \mathbf{R}$ for each j and subsequences of u_n and v_n satisfying

$$\widetilde{u}_{j,n} \longrightarrow \log \frac{1}{\left(1 + \frac{|\widetilde{x}|^2}{8}\right)^2}, \quad \widetilde{v}_{j,n} \longrightarrow \frac{\boldsymbol{a}_j \cdot \widetilde{x}}{8 + |x|^2} + b_j \frac{8 - |\widetilde{x}|^2}{8 + |\widetilde{x}|^2},$$

locally uniformly. We shall show $\boldsymbol{a}_j = \mathbf{0}$ and $b_j = 0$.

The proof is divided into 3 steps:

Step 1: We show the following asymptotic behavior for (a subsequence of) v_n:

$$\frac{v_n}{\lambda_n^{\frac{1}{2}}} \longrightarrow 2\pi \sum_{j=1}^m C_j \boldsymbol{a}_j \cdot \nabla_y G(x, \kappa_j) \tag{9}$$

locally uniformly in $\overline{\Omega} \setminus \cup_{j=1}^m B_{2\bar{R}}(\kappa_j)$, where $C_j > 0$ is some constant.

Step 2: Using the fact that $\mathscr{S}$ is a non-degenerate critical point of H^m, we show $\boldsymbol{a}_j = \mathbf{0}$ for every j.

Step 3: We show $b_j = 0$ for every j and consequently we show the uniform convergence $v_n \longrightarrow 0$ in Ω, which contradicts $\|v_n\|_{L^\infty(\Omega)} \equiv 1$.

Here we remark further on Step 2, which is based on the simple observation that

$$-\Delta u_{x_i} = \lambda e^u u_{x_i}, \tag{10}$$

holds for every solution u of (1), that is, $u_{x_i} = \frac{\partial u}{\partial x_i}$ is always a solution of (8) except for the boundary condition. Then using the Green identity, we get

$$\int_{\partial B_R(\kappa_j)} \left(\frac{\partial}{\partial \nu}(u_n)_{x_i} v_n - (u_n)_{x_i} \frac{\partial}{\partial \nu} v_n \right) d\sigma = 0. \tag{11}$$

for every κ_j and sufficiently small $R\,(> 2\bar{R}) > 0$. From the know asymptotic behaviors of u_n and v_n, we are able to see that the limit of the above identity is a linearly combination of the integration

$$I_{ij} := \int_{\partial B_R(z_1)} \left\{ \frac{\partial}{\partial \nu_x} G_{x_i}(x, z_2) G_{y_j}(x, z_3) - G_{x_i}(x, z_2) \frac{\partial}{\partial \nu_x} G_{y_j}(x, z_3) \right\} d\sigma_x.$$

We are able to calculate this and get some *localized* versions of the known integral identity for the Green function:

$$I_{ij} = I_{ij}(z_1, z_2, z_3) = \begin{cases} 0 & (z_1 \neq z_2,\ z_1 \neq z_3), \\ \frac{1}{2} R_{x_i x_j}(z_1) & (z_1 = z_2 = z_3), \\ G_{x_i y_j}(z_1, z_3) & (z_1 = z_2,\ z_1 \neq z_3), \\ G_{x_i x_j}(z_1, z_2) & (z_1 \neq z_2,\ z_1 = z_3). \end{cases} \tag{12}$$

Collecting the limit of (11) for all $j = 1, \cdots, m$, we get

$$0 = 16\pi^2 \,\mathrm{Hess} H^m|_{(x_1,\cdots,x_m)=(\kappa_1,\cdots,\kappa_m)} \,{}^t(C_1 \boldsymbol{a}_1, \cdots, C_m \boldsymbol{a}_m).$$

This gives $\boldsymbol{a}_j = \mathbf{0}$ from the assumption that $\mathrm{Hess} H^m$ is invertible at $\mathscr{S}$.

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