Blow-up analysis and optimal Trudinger-Moser inequalities for some mean field equations in statistical hydrodynamics

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1 Introduction

In recent years, several mean field equations have been derived in order to describe two-dimensional turbulence, following Onsager's statistical mechanics approach [4], see also [1, 8].

We consider a class of such equations containing a *probability measure* $\mathcal{P}$, which describes the distribution of the intensity and of the orientation of the vortices. The following equation was derived by Sawada and Suzuki in [7]:

$$-\Delta v = \lambda \int_I \alpha \left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}\, dx} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\alpha) \qquad \text{on } \Omega, \tag{1}$$

where Ω is a compact two-dimensional orientable Riemannian manifold without boundary, $I = [-1, 1]$, $\mathcal{P} \in \mathcal{M}(I)$ is a Borel measure which determines the distribution of the signed vortex circulation, $v \in H^1(\Omega)$ is a function normalized by $\int_\Omega v = 0$, $\lambda > 0$ is a constant related to the statistical temperature. Equation (1) admits a variational formulation. Indeed it is the Euler-Lagrange equation of the functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \int_I \log \left(\int_\Omega e^{\alpha v} \right) \mathcal{P}(d\alpha),$$

defined for $v \in H^1(\Omega)$, $\int_\Omega v = 0$.

A similar equation was derived by Neri [2], under the assumption that the vortex circulation is a random variable with distribution $\mathcal{P}$:

$$-\Delta v = \lambda \frac{\int_I \alpha(e^{\alpha v} - \frac{1}{|\Omega|}\int_\Omega e^{\alpha v} dx)\,\mathcal{P}(d\alpha)}{\iint_{I\times\Omega} e^{\alpha v}\,\mathcal{P}(d\alpha)dx} \qquad \text{on } \Omega. \tag{2}$$

Neri's equation is the Euler-Lagrange equation for the functional:

$$\mathcal{K}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\iint_{I\times\Omega} e^{\alpha v}\,dx\mathcal{P}(d\alpha)\right).$$

We address the mathematical issues of the blow-up analysis of concentrating sequences of solutions to equation (1) and equation (2), as well as the optimal value of λ which ensures boundedness from below of J_λ and K_λ. The blow-up analysis turns out to be quite similar for the two equations. On the other hand, the best constants for boundedness below behave differently with respect to $\mathcal{P}$.

We are thus led to ask the following.

Question. Can the different behavior of the variational functionals be used as a criterion to select the more consistent model?

These results are obtained in collaboration with H. Ohtsuka, T. Suzuki and G. Zecca and is contained in [3, 5, 6].

References

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