Mean field equation for vortex filament systems -derivation, dual variational structure, existence of the solution-

Ken Sawada

Meteorological College, Kashiwa-shi Chiba 277-0852, Japan

We consider vortex filament systems composed of nearly parallel vortex filaments in a columnar region with cross section $\Omega \subset \mathbb{R}^2$, in the frame of P-broken path model with the periodic boundary condition. From the fact that they are described by Hamiltonian systems, several mean field equations have been derived [1, 3]. For the system with a probability measure $\mathcal{P}(d\alpha)$ on $[-1,1]$ which describes the number density of the filaments with a certain circulation $\alpha^m \in [-1,1]$:

$$\mathcal{P}(d\alpha) \quad = \quad \sum_{m=1}^{M} n^m \delta_{\alpha^m}(d\alpha), \quad \sum_{m=1}^{M} n^m = 1,$$

the following mean field equation is derived.

$$\begin{cases} -\Delta v_i = \lambda \displaystyle\int_{[-1,1]} \alpha \frac{K_i e^{\alpha v_i}}{\int_\Omega K_i e^{\alpha v_i} dx_i} \mathcal{P}(d\alpha) & \text{in } \Omega \\ v_i = 0 & \text{on } \partial\Omega \end{cases} \quad i = 1, \cdots, P, \tag{1}$$

where

$$\begin{aligned} K_i &= K_i(x_i, \lambda) \\ &= \int_{\Omega^{P-1}} e^{\gamma \sum_{j=1}^{P} |x_{j+1} - x_j|^2} e^{\alpha \sum_{j=1, j\neq i}^{P} v_j(x_j)} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_P \\ & \quad x_{P+1} = x_1, \ \gamma = \frac{\lambda S}{8\pi}, \\ & \quad S : \text{structure parameter}, \ \lambda : \text{transformed inverse temperature}. \end{aligned}$$

The vortex filament system is thought as an extension to three dimensional case of the point vortex system. In the point vortex system, the quantization of blowup mechanism [2], the existence of the solution [4] of the mean field equations, and the dual variational structure [5, 6] have been investigated. In more detail, Lagrangian $L(v,u)$ on $X \times X^*$, ($X = H_0^1(\Omega)$, $X^* = H^{-1}(\Omega)$) for the mono-circulation point vortex system is defined by

$$\begin{aligned} L(v,u) \quad &= \quad \int_\Omega u(\log u - 1)dx - \lambda(\log\lambda - 1) \\ & \qquad + \frac{1}{2}\|\nabla v\|_2^2 - \langle v, u\rangle + 1_{D(F^*)} \\ & \qquad D(F^*) = \{u \in X \mid u \geq 0, \int_\Omega u = \lambda\}, \end{aligned}$$

and the Toland duality

$$\inf_{(v,u)\in X\times X^*} L(v,u) = \inf_{u\in X^*} J^*(u) = \inf_{v\in X} J(v) \tag{2}$$

is satisfied, where

$$\begin{aligned} J(v) &= \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v dx \\ J^*(u) &= \int_\Omega u(\log u - 1)dx - \lambda(\log\lambda - 1) - \frac{1}{2}\langle(-\Delta_D)^{-1}u, u\rangle + 1_{D(F^*)}. \end{aligned}$$

The mean field equations for this sysytem are derived as Euler-Lagrange equations from the variational functionals J and J^*. Here, we note that v and u are related to the stream function and vorticity, respectively.

For the mono circulation vortex filament system ($\mathcal{P}(d\alpha) = \delta_{+1}(d\alpha)$), the mean field equation

$$\begin{cases} -\Delta v_i = \dfrac{K_i e^{v_i}}{\int_\Omega K_i e^{v_i} dx} & \text{in } \Omega \\ v_i = 0 & \text{on } \partial\Omega, \end{cases} \quad i = 1, \cdots, P \tag{3}$$

has a variational formulation, and the variational functional for $v_1, \cdots, v_P \in H_0^1(\Omega)$ is given by

$$\begin{aligned} &J(v_1, \cdots, v_P) \\ &= \frac{1}{2}\sum_{i=1}^P \|\nabla_i v_i\|_2^2 - \lambda \log \int_{\Omega^P} e^{\sum_{i=1}^P (\gamma|x_{i+1}-x_i|^2 + v_i(x_i))} dx_1 \cdots dx_P. \end{aligned}$$

We obtain the following results.

Theorem 1 *This system is equipped with a dual variational structure, as in the case of the point vortex system.*

Theorem 2 *There exists a global minimizer for variational functional J and the classical solution to the mean field equation for $v_1, \cdots, v_P$, if $\lambda \in (0, 8\pi)$.*

This study is a joint work with Prof. T. Suzuki.

References

[1] R. Klein, A.J. Majda, and K. Damodaran, *Simplified equation for the iteration of nearly parallel vortex filaments*, J. Fluid Mech. **228** (1995) 201-248.

[2] N. Nagasaki and T. Suzuki, *Asymptotic analysis for two-dimensional elliptic eigenvalue problem with exponentially dominated nonlinerities*, Asymptot. Anal. **3** (1990) 173-188.

[3] K. Sawada and T. Suzuki, *Derivation of the equilibrium mean field equation of point vortex system and vortex filament system*, Theoretical and Applied Mechanics Japan **56** (2007) 285-290.

[4] T. Suzuki, *Global analysis for a two-dimensional elliptic eigenvalue problem with the exponential nonlinearity*, Ann. Inst. H. Poincaré Anal. Non Linéaire **9** No.4 (1992) 367-398.

[5] T. Suzuki, *Free Energy and Self-Interacting Particles*, Birkhäuser, Boston, 2005.

[6] T. Suzuki, *Mean Field Theories and Dual Variation*, Atlantis Press, Amsterdam-Paris, 2008.