Propagation of the analyticity for the solution to the Euler equations with non-decaying initial velocity [1]

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In this talk, we consider the Euler equations in $\mathbb{R}^n$ with $n \geqslant 2$, describing the motion of perfect incompressible fluids,

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{E}$$

where the unknown functions $u = u(x,t) = (u^1(x,t), \dots, u^n(x,t))$ and $p = p(x,t)$ denote the velocity field and the pressure of the fluid, respectively, while $u_0 = u_0(x) = (u_0^1(x), \dots, u_0^n(x))$ denotes the given initial velocity field.

The purpose of this talk is to show the propagation properties of the real analyticity in spatial variables for the solution to (E) with non-decaying initial velocity. Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz class of all rapidly decreasing functions, and let $\mathscr{S}'(\mathbb{R}^n)$ be the space of all tempered distributions. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of all positive integers. We first recall the definition of the Littlewood-Paley operators. Let $\{\varphi_j\}_{j=0}^\infty$ be a sequence in $\mathscr{S}(\mathbb{R}^n)$ satisfying the following properties (i), (ii) and (iii) :

(i) $\operatorname{supp} \widehat{\varphi_0} \subset \left\{ \xi \in \mathbb{R}^n \;\middle|\; |\xi| \leqslant 2 \right\}$, $\quad \operatorname{supp} \widehat{\varphi_j} \subset \left\{ \xi \in \mathbb{R}^n \;\middle|\; 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1} \right\}$ if $j \in \mathbb{N}$.

(ii) For any multi-index $\alpha \in \mathbb{N}_0^n$ there exists a positive constant c_α such that

$$\sup_{j \in \mathbb{N}_0} \sup_{\xi \in \mathbb{R}^n} 2^{j|\alpha|} \left| \partial_\xi^\alpha \widehat{\varphi_j}(\xi) \right| \leqslant c_\alpha.$$

(iii) $\displaystyle\sum_{j=0}^\infty \widehat{\varphi_j}(\xi) = 1$ for all $\xi \in \mathbb{R}^n$.

Given $f \in \mathscr{S}'(\mathbb{R}^n)$, we denote $\Delta_j f := \varphi_j * f$ for $j \in \mathbb{N}_0$, where $*$ denotes the convolution operator. Then, we define the Besov spaces $B^1_{\infty,1}(\mathbb{R}^n)$ by the following definition.

Definition 1. The Besov space $B^1_{\infty,1}(\mathbb{R}^n)$ is defined to be the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^1_{\infty,1}} := \sum_{j=0}^\infty 2^j \|\Delta_j f\|_{L^\infty} < \infty.$$

Note that $B^1_{\infty,1}(\mathbb{R}^n)$ is a Banach space with its norm $\|\cdot\|_{B^1_{\infty,1}}$. It is easy to see the continuous embedding $H^m(\mathbb{R}^n) \hookrightarrow B^1_{\infty,1}(\mathbb{R}^n)$ if $m > n/2+1$. Moreover, $B^1_{\infty,1}(\mathbb{R}^n)$ contains some non-decaying functions, for example, $e^{ia\cdot x} \in B^1_{\infty,1}(\mathbb{R}^n) \setminus H^m(\mathbb{R}^n)$ for all wavevector $a \in \mathbb{R}^n$.

For the local-in-time existence and uniqueness of solutions for (E), Kato [2] proved that for the given initial velocity field $u_0 \in H^m(\mathbb{R}^n)^n$ with $m > n/2+1$, there exist $T = T(\|u_0\|_{H^m})$ and a unique solution u of (E) in the class $C([0,T]; H^m(\mathbb{R}^n)^n)$. Pak and Park [4] extended this result to the Besov spaces $B^1_{\infty,1}(\mathbb{R}^n)$. It has already been known that Kato's solution is

[1] Joint work with Professor Okihiro Sawada (Gifu University).

real analytic in spatial variables if $u_0 \in C^\omega(\mathbb{R}^n)^n$, see Alinhac and Métivier [1] and Kukavica and Vicol [3]. In this talk, we prove the propagation of analyticity of Pak-Park's solutions. In particular, we give an improvement for the estimate for the size of the radius of convergence of Taylor's expansion.

Before stating our result about the analyticity, we set notation. For $k \in \mathbb{N}_0$, put $m_k := ck!/(k+1)^2$ where c is a positive constant such that one has

$$\sum_{0 \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} m_{|\beta|} m_{|\alpha-\beta|} \leqslant m_{|\alpha|}, \quad \alpha \in \mathbb{N}_0^n,$$

$$\sum_{0 < \beta \leqslant \alpha} \binom{\alpha}{\beta} m_{|\beta|-1} m_{|\alpha-\beta|+1} \leqslant |\alpha| m_{|\alpha|}, \quad \alpha \in \mathbb{N}_0^n \setminus \{0\}^n.$$

For example, it suffices to take $c \leqslant 1/16$. Our result on the propagation of the analyticity now reads:

Theorem 2. *Let $u_0 \in B^1_{\infty,1}(\mathbb{R}^n)^n$ with $\operatorname{div} u_0 = 0$, and let $u \in C([0,T]; B^1_{\infty,1}(\mathbb{R}^n)^n)$ be the solution of* (E)*. Suppose that $u_0 \in C^\omega(\mathbb{R}^n)^n$ in the following sense : there exist positive constants K_0 and ρ_0 such that*

$$\|\partial_x^\alpha u_0\|_{B^1_{\infty,1}} \leqslant K_0 \rho_0^{-|\alpha|} m_{|\alpha|}$$

for all $\alpha \in \mathbb{N}_0^n$. Then, $u(\cdot,t) \in C^\omega(\mathbb{R}^n)^n$ for all $t \in [0,T]$ and satisfies the following estimate : there exist positive constants $K = K(n, K_0)$, $L = L(n, K_0)$ and $\lambda = \lambda(n)$ such that

$$\|\partial_x^\alpha u(\cdot,t)\|_{B^1_{\infty,1}} \leqslant K \left(\frac{\rho_0}{L}\right)^{-|\alpha|} m_{|\alpha|} (1+t)^{\max\{|\alpha|-1,0\}} \exp\left\{\lambda|\alpha| \int_0^t \|u(\cdot,\tau)\|_{B^1_{\infty,1}} d\tau\right\} \tag{1}$$

for all $\alpha \in \mathbb{N}_0^n$ and $t \in [0,T]$.

Remark 3. (i) Since K, L and λ do not depend on T, (1) gives a grow-rate estimate for large time behavior of Pak-Park's solutions provided $\int_0^t \|u(\tau)\|_{B^1_{\infty,1}} d\tau$ is uniformly bounded in time.

(ii) From (1), one can derive the estimate for the size of the uniform analyticity radius of the solutions as follows :

$$\liminf_{|\alpha|\to\infty} \left(\frac{\|\partial_x^\alpha u(t)\|_{L^\infty}}{\alpha!}\right)^{-\frac{1}{|\alpha|}} \geqslant \frac{\rho_0}{L}(1+t)^{-1} \exp\left\{-\lambda \int_0^t \|u(\tau)\|_{B^1_{\infty,1}} d\tau\right\}.$$

Recently, Kukavica and Vicol [3] considered the vorticity equations of (E) in $H^s(\mathbb{T}^3)^3$ with $s > 7/2$, and obtained the following estimate for uniform analyticity radius :

$$\liminf_{|\alpha|\to\infty} \left(\frac{\|\partial_x^\alpha \operatorname{rot} u(t)\|_{L^\infty}}{\alpha!}\right)^{-\frac{1}{|\alpha|}} \geqslant \rho(1+t^2)^{-1} \exp\left\{-\lambda \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\right\}$$

with some $\rho := \rho(r, \operatorname{rot} u_0)$ and $\lambda = \lambda(r)$. Hence our result is an improvement of the previous analyticity-rate in the sense that $(1+t^2)^{-1}$ is replaced by $(1+t)^{-1}$, and clarifies that $\rho = \rho_0/L$.

References

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