Uniqueness of almost periodic-in-time solutions to Navier-Stokes equations in unbounded domains

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We consider a viscous incompressible fluid in 3-dimensional unbounded domains Ω. The motion of such a fluid is governed by the Navier-Stokes equations:

$$\text{(N-S)} \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f, & t \in \mathbb{R}, \quad x \in \Omega, \\ \nabla \cdot u = 0, & t \in \mathbb{R}, \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, & t \in \mathbb{R}, \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $p = p(x,t)$ denote the velocity vector and the pressure, respectively, of the fluid at the point $(x,t) \in \Omega \times \mathbb{R}$. Here f is a given external force. We present a uniqueness theorem for almost periodic-in-time solutions to the Navier-Stokes equations in 3-dimensional unbounded domains. Thus far, uniqueness of almost periodic-in-time solutions to the Navier-Stokes equations in unbounded domain, roughly speaking, is known only for a small almost periodic-in-time solution in $BC(R; L^3_w)$ within the class of solutions which have sufficiently small $L^\infty(L^3_w)$-norm. We show that a small almost periodic-in-time solution in $BC(R; L^3_w \cap L^{6,2})$ is unique within the class of all almost periodic-in-time solutions in $BC(R; L^3_w \cap L^{6,2})$. Here L^3_w and $L^{p,q}$ denote the weak-L^3 space and the Lorentz space, respectively. We note that $L^{p,\infty}$ is equivalent to L^p_w.

We first introduce the definition of almost periodic functions.

Definition. Let B be a Banach space and $f \in C(\mathbb{R}; B)$. Then f is called an almost periodic function in B if for all $\epsilon > 0$ there exists $L = L(\epsilon) > 0$ with the following property: For all $k \in \mathbb{Z}$, there exists $T_{\epsilon k} \in [-(k+1)L, -kL]$ such that

$$\sup_{t \in \mathbb{R}} \|f(t + T_{\epsilon k}) - f(t)\|_B \le \epsilon.$$

Now our main result reads as follows:

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain, the half-space $\mathbb{R}^3_+$, the whole space $\mathbb{R}^3$, a perturbed half-space, or an aperture domain with $\partial\Omega \in C^\infty$. Then, there exists an absolute constant $\delta > 0$ such that if u and v are almost periodic-in-time in $L^{3,\infty}$ and mild solutions to (N-S) for the same external force f, if*

$$u, v \in L^2_{uloc}(\mathbb{R}; L^{6,2}(\Omega)),$$

and

$$\sup_t \|u\|_{3,\infty} < \delta,$$

then $u = v$.

The proof of the our uniqueness theorem is based on the method of dual equations. In order to prove our main theorem, it suffices to show that $w := u - v$ identically equals 0. Since u and v are solutions to the Navier-Stokes equations, w satisfies

$$(U) \qquad \begin{cases} \partial_t w - \Delta w + w \cdot \nabla u + v \cdot \nabla w + \nabla p' = 0, \qquad t \in \mathbb{R},\ x \in \Omega, \\ \quad \nabla \cdot w = 0, \qquad t \in \mathbb{R},\ x \in \Omega, \\ \quad w|_{\partial\Omega} = 0. \end{cases}$$

Hence, if Ω is a bounded domain and if u, v belong to the Leray-Hopf class, under the hypotheses of Theorem 1, the usual energy method and the Poincaré inequality yield $\|w(t)\|_2^2 \le e^{-(t-s)}\|w(s)\|_2^2$ for $t > s$. Consequently, in the case of *bounded* domains, Theorem 1 is obvious. In the case where Ω is an *unbounded* domain, u and v do not belong to the energy class in general and the Poincaré inequality does not hold in general. Hence, since we cannot use the energy method, we will use the method of dual equations.

The dual equations of the above system (U) are as follows.

$$\text{(D)} \qquad \begin{cases} -\partial_t \psi - \Delta \psi - \displaystyle\sum_{i=1}^{3} u^i \nabla \psi^i - v \cdot \nabla \psi + \nabla \pi &= F, \quad t < 0,\ x \in \Omega, \\ \nabla \cdot \psi &= 0, \quad t < 0,\ x \in \Omega, \\ \psi|_{\partial\Omega} &= 0. \end{cases}$$

By using the energy estimate, for all almost periodic functions F in $L^2(\Omega) \cap L^{6/5}(\Omega)$, we construct a sequence of weak solutions of (D) having a property similar to that of almost periodic functions.

References

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