Analytical derivation of diffusion coefficient of two-dimensional point vortex system with Klimontovich formalism

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An implicit diffusion effect involved in the two-dimensional (2D) point vortex system is investigated analytically.

Two-dimensional point vortex,

$$\omega_z(\boldsymbol{r},t) = \sum_i \Omega_i \delta(\boldsymbol{r} - \boldsymbol{r}_i), \tag{1}$$

is a formal solution for the inviscid, incompressible 2D Euler equation in the vorticity equation form,

$$\frac{\partial \omega_z(\boldsymbol{r},t)}{\partial t} + \nabla \cdot (\omega_z(\boldsymbol{r},t)\boldsymbol{u}(\boldsymbol{r},t)) = 0, \tag{2}$$

where ω_z is the vorticity in $x-y$ plane, Ω_i and $\boldsymbol{r}_i$ are the circulation and the position vector of the $i-$th point vortex. The Dirac delta function is denoted by $\delta(\boldsymbol{r})$. The solution (1) is an accumulation of singular points represented by the delta function and evolution of the points are achieved by the long-range interactions between themselves. It is likely that the pointlike discrete distribution yields a diffusive effect in analogy with the other collisional $N-$body systems. Thus, we presumed that a diffusive effect should be involved in the 2D point vortex system.

We introduce two scales, "microscopic" and "macroscopic". The well-known point vortex solution (1) is regarded as a microscopic solution of a microscopic Euler equation:

$$\frac{\partial \hat{\omega}_z(\boldsymbol{r},t)}{\partial t} + \nabla \cdot (\hat{\omega}_z(\boldsymbol{r},t)\hat{\boldsymbol{u}}(\boldsymbol{r},t)) = 0, \tag{3}$$

where a hat means a quantity is a microscopic one and the corresponding macroscopic quantity is obtained by an ensemble average $\langle \cdot \rangle$:

$$\omega_z(\boldsymbol{r},t) \equiv \langle \hat{\omega}_z(\boldsymbol{r},t) \rangle. \tag{4}$$

Note that the left-hand side is the macroscopic quantity. To derive the effective diffusion term, the Klimontovich formalism is utilized:

$$\hat{\omega}_z(\boldsymbol{r},t) = \omega_z(\boldsymbol{r},t) + \delta\hat{\omega}_z(\boldsymbol{r},t). \tag{5}$$

The microscopic vorticity consists of the macroscopic vorticity and a fluctuation. Substituting Eq. (5) into Eq. (3) and averaging the equation (neglecting the first order terms of the fluctuation), we obtain a macroscopic vorticity equation:

$$\frac{\partial \omega_z(\boldsymbol{r},t)}{\partial t} + \nabla \cdot (\omega_z(\boldsymbol{r},t)\boldsymbol{u}(\boldsymbol{r},t)) = -\nabla \cdot \langle \delta\hat{\boldsymbol{u}}(\boldsymbol{r},t)\delta\hat{\omega}_z(\boldsymbol{r},t) \rangle. \tag{6}$$

Introducing the linearlized equation for the fluctuation, the right-hand side of Eq. (6) can be rewritten explicitly:

$$\frac{\partial \omega_z(\boldsymbol{r},t)}{\partial t} + \nabla \cdot (\boldsymbol{u}(\boldsymbol{r},t)\boldsymbol{u}(\boldsymbol{r},t)) = -\nabla \cdot (\vec{\eta} \cdot \nabla \omega_z), \tag{7}$$

$$\vec{\eta} = \int_{-\infty}^{t} d\tau \langle \delta\hat{\boldsymbol{u}}(\boldsymbol{r},t)\delta\hat{\boldsymbol{u}}(\boldsymbol{r} - (t-\tau)\boldsymbol{u}, \tau) \rangle. \tag{8}$$

The obtained diffusion term includes the position correlation and is an extension of the well-known Green-Kubo formula that includes the time correlation only.

This result suggests that the point vortex method is suitable for a simulation of a high Reynolds number system that has finite but very small viscosity.